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LETTER TO THE EDITOR

Boundary conditions for Nicolai maps

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Abstract. It is shown that Nicolai's theorem does not generally apply on spaces with boundaries unless the supersymmetry algebra has a subalgebra whose bosonic generators preserve the boundary. In this case, the unbroken symmetries generated by the boundary-preserving subalgebra are sufficient to ensure the existence of a Nicolai map, provided that any boundary conditions are invariant under this subalgebra.

In 1980 Nicolai showed that for any supersymmetric action there is a transformation of the bosonic variables which renders the bosonic part of the action free, and whose Jacobian is exactly cancelled by the integration of the fermionic variables [1]. Thus, any supersymmetric theory can be transformed to a free bosonic theory. Such a transformation is called a Nicolai map.

The simplest example of a Nicolai map is Langevin's equation, which establishes a functional dependence between two stochastic variables: the position of a Brownian particle, and the white noise representing the random forces which drive it. The behaviour of the Brownian particle can be described by Witten's supersymmetric quantum mechanics [2]. If one starts with a path integral description of this model and integrates out the fermions, then changing variables via the Langevin equation yields a description of the generating noise process. The action functional for the noise is found to be quadratic, indicating that the noise is Gaussian.

The Langevin equation is typical of Nicolai maps in that it takes the form of a differential equation relating the two sets of bosonic variables. It would appear that such a map could be made invertible by imposing a certain number of boundary conditions on the bosonic variables in the supersymmetric theory. Unfortunately, local boundary conditions are not respected by the full supersymmetry group, which contains translations normal to the boundary. The imposition of such conditions would therefore break the supersymmetry and invalidate the usual proof of Nicolai's theorem†.

Nicolai's theorem rests on a result due to Zumino [3], that the vacuum energy of any supersymmetric quantum theory vanishes to all orders of perturbation theory. In other words, the normalization of the fuctional integral is unaffected by any supersymmetric perturbation of the action. Zumino's original proof fails when the supersymmetry is broken by the presence of boundaries; however, it is shown below that the desired result can be recovered if the supersymmetry has some subalgebra whose bosonic generators preserve the boundary. In this case, a partial supersymmetry can be restored by adding a total divergence to the super-Lagrangian.

† This problem does not arise for periodic conditions, which are interpreted more naturally as continuity requirements on multiply-connected spaces than as boundary conditions.

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Thus, when boundaries are present, the proof of Nicolai's theorem requires the existence of a boundary-preserving subalgebra. It is also necessary that any boundary conditions must be invariant under this subalgebra. This invariance places quite strict requirements on the form of admissible boundary conditions.

Zumino's result relies on a demonstration that the expectation value of a supersymmetric Lagrangian vanishes in a supersymmetric state. This argument is summarized below.

We work in a superspace parametrized by commuting coordinates x^μ ($\mu = 1, \ldots, n$) and anticommuting coordinates θ^a ($a = 1, \ldots, D$). The model has a super-Lagrangian

$$\begin{aligned}\mathscr{L}(x, \theta) = A(x) + \theta^a B_a(x) + \frac{1}{2}\theta^a\theta^b C_{ab}(x) + \ldots\\ + \frac{1}{2}\frac{\partial}{\partial\theta^a}\frac{\partial}{\partial\theta^b}(\theta^1\theta^2\ldots\theta^D)J^{ab}(x) + \frac{\partial}{\partial\theta^a}(\theta^1\theta^2\ldots\theta^D)K^a(x)\\ + (\theta^1\theta^2\ldots\theta^D)L(x)\end{aligned} \tag{1}$$

which depends on some superfield $\Phi(x, \theta)$. In the case of simple supersymmetry the fermionic generators have the form

$$Q_a = \frac{\partial}{\partial\theta^a} + (C\gamma^\mu)_{ab}\theta^b\partial_\mu \tag{2}$$

where C is the charge conjugation matrix and γ^μ are appropriate gamma matrices. The variation of the K^a component of the super-Lagrangian under an infinitesimal supersymmetry transformation $\mathscr{L} \mapsto \mathscr{L}_\epsilon = (1 + \epsilon^b Q_b)\mathscr{L}$ is then

$$\delta_\epsilon K^a = (-1)^D \epsilon^b[(C\gamma^\mu)_{bc}\partial_\mu J^{ac} - \delta^a_b L]. \tag{3}$$

Taking expectation values $\langle\,\rangle_{\mathscr{E}}$ over an ensemble (or quantum state) $\mathscr{E}$ which is invariant under supersymmetry, we find that

$$0 = \delta_\epsilon\langle K^a\rangle_{\mathscr{E}} = (-1)^D\epsilon^b[(C\gamma^\mu)_{bc}\partial_\mu\langle J^{ac}\rangle_{\mathscr{E}} - \delta^a_b\langle L\rangle_{\mathscr{E}}]. \tag{4}$$

But the supersymmetry algebra includes the generators ∂_μ, and so the invariance of $\mathscr{E}$ implies that $\langle J^{ac}\rangle_{\mathscr{E}}$ is constant. It follows that the expectation value of the Lagrangian vanishes identically;

$$\langle L\rangle_{\mathscr{E}} = 0. \tag{5}$$

Note that non-invariant boundary conditions on Φ would break the assumed supersymmetry of the ensemble $\mathscr{E}$ and thereby invalidate the conclusion (5). The argument would also fail if the weight of each field configuration in $\mathscr{E}$—i.e. the functional measure—was not strictly invariant under supersymmetry transformations. In particular, if the weight is the exponential of an action, then the variation of the latter must vanish exactly.

Suppose now that we have a super-Lagrangian consisting of two parts,

$$\mathscr{L}_g[\Phi] = \mathscr{L}_0[\Phi] + g\mathscr{L}_{\text{int}}[\Phi] \tag{6}$$

each transforming separately as a superfield. The coefficient g is thought of as a coupling constant. If the action of a configuration Φ is given by the integral

$$S_g[\Phi] \equiv \int \mathrm{d}^n x\, \mathrm{d}^D\theta\, \mathscr{L}_g = \int \mathrm{d}^n x\, L_g \tag{7}$$

then we define a regularized connected vacuum functional U_g with

$$e^{-U_g}=\frac{\int d[\Phi]\, e^{-S_g[\Phi]}}{\int d[\Phi]\, e^{-S_0[\Phi]}}. \tag{8}$$

One now has

$$\frac{dU}{dg}=\int d^n x \langle L_{\rm int}\rangle_g \tag{9}$$

where the subscript g indicates that the expectation value is taken over an ensemble in which each superfield configuration Φ has weight $e^{-S_g[\Phi]}$. In the absence of boundaries, S_g is invariant under supersymmetry transformations and so (5) implies that the expectation value of $L_{\rm int}$ vanishes identically. It then follows that U_g is independent of g, and so

$$\int d[\Phi]\, e^{-S_g[\Phi]}=\int d[\Phi]\, e^{-S_0[\Phi]} \tag{10}$$

for all g. This identity is used in the proof of Nicolai's theorem.

It is apparent that the argument used to obtain (10) simply does not hold if the supersymmetry is broken by boundaries. It would therefore seem that Nicolai's theorem must be abandoned in this case. However, we will see that under certain conditions it is still valid.

If the supersymmetry algebra has a subalgebra whose bosonic generators preserve the boundary, it turns out the theory can be made invariant under this subalgebra by the addition of a total divergence to the super-Lagrangian. By integrating out the dependence on certain coordinates one then obtains a 'reduced' supersymmetry in a space of lower dimension [4]. Zumino's argument can then be used to recover identity (10), as required for Nicolai's theorem.

Indeed, suppose that there is such a subalgebra. Without loss of generality we take its generators to be $\{\partial_{\hat\mu}: \hat\mu=1,\ldots,\hat n\}$ and $\{Q_{\hat a}: \hat a=1,\ldots,\hat D\}$, where $\hat n<n$ and $\hat D<D$. In order that the subalgebra closes [4], we require that

$$(C\gamma^{\bar\mu})_{\hat a\hat b}=0 \qquad \bar\mu=\hat n+1,\ldots,n \quad \hat a,\hat b=1,\ldots \hat D. \tag{11}$$

In what follows, hatted Greek indices range from 1 to $\hat n$ while hatted Latin indices range from 1 to $\hat D$. Similarly, barred Greek indices range from $\hat n+1$ to n, while barred Latin indices range from $\hat D+1$ to D.

If $\mathscr{L}$ is replaced by the modified super-Lagrangian

$$\mathscr{L}'=\exp(\theta^{\bar a}(C\gamma^{\mu})_{\bar a\bar b}\theta^{\bar b}\partial_\mu)\mathscr{L} \tag{12}$$

then the action functional

$$S[\Phi]=\int d^n x\, d^D\theta\, \mathscr{L}'[\Phi] \tag{13}$$

is invariant under the subalgebra [4]. In fact $\mathscr{L}$ and $\mathscr{L}'$ only differ by a total divergence, so that all we have done is to add a boundary correction to the usual action.

If Φ obeys constraints—in particular, boundary conditions—which are invariant under the boundary-preserving subalgebra, then the set of superfield configurations which contribute to the functional integral will also be invariant. The invariance of the action—and hence the functional measure—then ensures that the statistical ensemble (or quantum state) is itself invariant under the boundary-preserving subalgebra.

We now define a reduced super-Lagrangian $\mathscr{L}^{\mathrm{red}}(x^{\hat{\mu}}, \theta^{\hat{a}})$ by integrating out the dependence of $\mathscr{L}'$ on the coordinates $(x^{\bar{\mu}}, \theta^{\bar{a}})$;

$$\mathscr{L}^{\mathrm{red}}(x^{\hat{\mu}}, \theta^{\hat{a}}) \equiv \int \prod_{\bar{\mu}=\hat{n}+1}^{n} \mathrm{d}x^{\bar{\mu}} \prod_{\bar{a}=\hat{D}+1}^{D} \mathrm{d}\theta^{\bar{a}}\, \mathscr{L}'(x^{\mu}, \theta^{a}). \tag{14}$$

Under an infinitesimal transformation $\mathscr{L} \mapsto \mathscr{L}_{\epsilon} = (1+\varepsilon^{\hat{a}} Q_{\hat{a}})\mathscr{L}$ the variation of the reduced super-Lagrangian is found to be

$$\delta_{\epsilon}\mathscr{L}^{\mathrm{red}} = \epsilon^{\hat{a}} \hat{Q}_{\hat{a}} \mathscr{L}^{\mathrm{red}} \tag{15}$$

where

$$\hat{Q}_{\hat{a}} \equiv \frac{\partial}{\partial \theta^{\hat{a}}} + (C\gamma^{\hat{\mu}})_{\hat{a}\hat{b}}\theta^{\hat{b}}\partial_{\hat{\mu}} \tag{16}$$

are the fermionic generators of a 'reduced' supersymmetry which acts in $\hat{n}$ dimensions [4]. These obey exactly the same anticommutation relations as the $Q_{\hat{a}}$.

The boundary-preserving subalgebra can therefore be represented by the action of the reduced supersymmetry generators $\hat{Q}_{\hat{a}}, \partial_{\hat{\mu}}$ on $\mathscr{L}^{\mathrm{red}}$. Furthermore, S can be expressed as the integral of $\mathscr{L}^{\mathrm{red}}$ over the remaining coordinates $(x^{\hat{\mu}}, \theta^{\hat{a}})$. If $\mathscr{L}^{\mathrm{red}}$ consists of two parts as in (6), then the invariance of the ensemble under the reduced supersymmetry allows us to apply Zumino's argument in $\hat{n}$ dimensions to recover the desired result (10). The proof of Nicolai's theorem then follows directly.

The existence of a boundary-preserving subalgebra is therefore sufficient to ensure the existence of a Nicolai map, provided that one uses the invariant action (13) and imposes only invariant boundary conditions. In the absence of such a subalgebra, however, there appears to be no way to restore even a partial supersymmetry, and neither Zumino's result nor Nicolai's theorem can be invoked.

The best known Nicolai map is the Langevin equation, which can be used to transform Witten's supersymmetric quantum mechanics [2] into a free bosonic theory describing the behaviour of the generating noise process. It has recently been observed [5] that Nicolai maps for this model exist only when the action includes corrections which restore part of the supersymmetry broken by the boundaries. This observation is simply explained by the preceding arguments, and provides a nice illustration of the general result.

The simplest version of supersymmetric quantum mechanics can be constructed from the super-Lagrangian

$$\mathscr{L} = \tfrac{1}{2}\delta_{ij}\bar{D}\Phi^{i} D\Phi^{j} + V(\Phi) \tag{17}$$

where Φ^{i} is a superfield defined on a superspace with coordinates $(t, \bar{\theta}, \theta)$;

$$\Phi^{i}(t, \bar{\theta}, \theta) = \phi^{i}(t) + \theta\bar{\psi}^{i}(t) + \bar{\theta}\psi^{i}(t) + \bar{\theta}\theta c^{i}(t). \tag{18}$$

The superderivatives D, $\bar{D}$ and the supersymmetry generators Q, $\bar{Q}$ have the form

$$D \equiv \frac{\partial}{\partial\bar{\theta}} - \theta\frac{\partial}{\partial t} \qquad \bar{D} \equiv \frac{\partial}{\partial\theta} - \bar{\theta}\frac{\partial}{\partial t} \qquad Q \equiv \frac{\partial}{\partial\bar{\theta}} + \theta\frac{\partial}{\partial t} \qquad \bar{Q} \equiv \frac{\partial}{\partial\theta} + \bar{\theta}\frac{\partial}{\partial t} \tag{19}$$

and obey the anticommutation relations

$$\{Q, \bar{Q}\} = -\{D, \bar{D}\} = 2\frac{\partial}{\partial t} \qquad \{Q, Q\} = \{\bar{Q}, \bar{Q}\} = \{D, D\} = \{\bar{D}, \bar{D}\} = 0. \tag{20}$$

Integrating out the dependence of $\mathscr{L}$ on the anticommuting coordinates, and eliminating the auxiliary field c by its field equation, one obtains the supersymmetric Lagrangian

$$L \equiv \int \mathrm{d}\theta\, \mathrm{d}\bar{\theta}\, \mathscr{L} = \tfrac{1}{2}\delta_{ij}\dot{\phi}^i\dot{\phi}^j + \tfrac{1}{2}\delta^{ij}V_{,i}V_{,j} + \tfrac{1}{2}\delta_{ij}(\bar{\psi}^i\dot{\psi}^j - \dot{\bar{\psi}}^i\psi^j) + V_{,ij}\bar{\psi}^i\psi^j. \tag{21}$$

In this example, the boundary is just a pair of points t_1, t_2. There are no bosonic symmetries which preserve the boundary, and so the only two boundary-preserving subalgebras are those spanned by Q and $\bar{Q}$ respectively.

Equation (12) tells us how the super-Lagrangian must be modified to restore invariance under each of these subalgebras. Integrating out the dependence on θ and $\bar{\theta}$ then yields the modified Lagrangians

$$L_+ \equiv L + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\,\delta_{ij}\bar{\psi}^i\psi^j + V(\phi)\right] = \frac{1}{2}(\dot{\phi}^i + V_{,i})^2 + \bar{\psi}^i\left(\delta_{ij}\frac{\mathrm{d}}{\mathrm{d}t} + V_{,ij}\right)\psi^j \tag{22a}$$

$$L_- \equiv L - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\,\delta_{ij}\bar{\psi}^i\psi^j + V(\phi)\right] = \frac{1}{2}(\dot{\phi}^i - V_{,i})^2 + \psi^i\left(\delta_{ij}\frac{\mathrm{d}}{\mathrm{d}t} - V_{,ij}\right)\bar{\psi}^j \tag{22b}$$

which are guaranteed to be invariant under Q and $\bar{Q}$ respectively. Our analysis shows that these are the only two forms of the Lagrangian which admit Nicolai maps.

Recall, however, that Nicolai's theorem can only be invoked if any boundary conditions are invariant under the appropriate subalgebra. Anticipating that the Nicolai maps will be first-order differential equations, we will want to specify an initial (or final) value of ϕ. The variation of ϕ under the action of Q is proportional to ψ, so to ensure that this boundary condition is Q-invariant we must also impose a Dirichlet condition on ψ. (No further conditions are needed, since ψ is itself Q-invariant.)

Thus, Q-invariant initial conditions suitable for the Lagrangian L_+ have the form

$$\phi(t_1) = \phi_1 \qquad \psi(t_1) = 0. \tag{23a}$$

Similarly, $\bar{Q}$-invariant initial conditions appropriate for L_- have the form

$$\phi(t_1) = \phi_1 \qquad \bar{\psi}(t_1) = 0. \tag{23b}$$

Our earlier analysis demonstrates that, for each of these initial conditions, the corresponding form of the Lagrangian (22) will admit a Nicolai map.

Indeed this is well known to be the case, although in the standard demonstration the supersymmetric theory is obtained from the bosonic theory rather than the other way around: One begins with a white noise process $\eta^i(t)$, defined over the interval $[t_1, t_2]$, with the probability distribution functional

$$\mathscr{P}[\eta(t)] = \exp\left\{-\tfrac{1}{2}\int_{t_1}^{t_2} \eta^i(t)\eta^j(t)\delta_{ij}\,\mathrm{d}t\right\}. \tag{24}$$

One then introduces a new stochastic process $\phi(t)$ which is related to $\eta(t)$ by a Langevin equation

$$\eta^i(t) = \frac{\mathrm{d}\phi^i}{\mathrm{d}t} \pm \delta^{ij}\frac{\partial V}{\partial\phi^i} \tag{25}$$

and a single boundary condition on ϕ, say $\phi(t_1) = \phi_1$.

The probability weight for a path $\phi(t)$ is now derived from (24) by a change of variables, which introduces the Jacobian factor

$$\mathrm{Det}\,\frac{\delta\eta}{\delta\phi} = \int \mathrm{d}[\bar{\chi}, \chi]\exp\left\{-\int_{t_1}^{t_2}\mathrm{d}t\int_{t_1}^{t_2}\mathrm{d}t'\,\bar{\chi}_i(t')\,\frac{\delta\eta^i(t')}{\delta\phi^j(t)}\,\chi^j(t)\right\}. \tag{26}$$

The anticommuting functions $\bar{\chi}^i(t)$ and $\chi^i(t)$ live in the same spaces as perturbations of the variables $\eta^i(t)$ and $\phi^i(t)$, and hence obey the same boundary conditions. Since $\phi(t_1)$ is fixed, $\chi(t_1)$ must vanish; however η and $\bar{\chi}$ are unconstrained.

Writing the integral (26) explicitly, with the anticommuting variables renamed ψ and $\bar{\psi}$ as appropriate, the probability weight for $\phi(t)$ is now just

$$\mathscr{P}[\phi(t)] = \int \mathrm{d}[\bar{\psi}, \psi] \exp\left\{-\int_{t_1}^{t_2} L_{\pm}[\phi, \bar{\psi}, \psi]\,\mathrm{d}t\right\} \tag{27}$$

where the signs are determined by the choice of signs in (25), the expressions for $L_{\pm}$ were given in (22), and all the variables obey exactly the initial conditions predicted in (23). In the present context, however, these conditions have not arisen from any supersymmetric considerations, but from the change of variables from $\eta(t)$ to $\phi(t)$.

The previous discussion showed that there are just two forms of the Lagrangian for supersymmetric quantum mechanics which admit Nicolai maps over a bounded time interval. It also predicted which boundary conditions were appropriate for each Lagrangian. We now see that the two known Nicolai maps for this model fulfil these predictions exactly.

Note that in the example considered above, the form of the Nicolai maps is almost obvious from the Lagrangians (22). In general, however, finding a Nicolai map is a very difficult problem. For example, if δ_{ij} is replaced by a superfield $G_{ij}(\Phi)$ in the super-Lagrangian (17), the supersymmetric Lagrangian will be quartic in the fermions and the fermionic integral will be much more difficult. (Nonetheless, various authors have succeeded in finding explicit Nicolai maps for this model [6].) In the general case, it is clear that a knowledge of the precise form of the action and the boundary conditions is likely to considerably simplify the search for a Nicolai map.

Nicolai's theorem says that for any supersymmetric theory there is a change of variables which transforms it to a free bosonic theory. In practice, however, most Nicolai maps have been obtained by the reverse process; starting with a bosonic theory, one performs a propitious change of variables to obtain the supersymmetric theory. From this perspective, the boundary conditions and corrections to the action appear to come out as an irrelevant by-product. Should one ever wish to find new Nicolai maps for given supersymmetric models, however, it would be very useful to know these in advance.

The arguments used above to determine the boundary conditions and corrections required for Nicolai's theorem can be applied to quite general models (provided that the supersymmetry algebra admits a boundary-preserving subalgebra). The exact form of the Lagrangian is obtained by projecting out the relevant component of the modified super-Lagrangian (12).

There is also a simple way of choosing invariant boundary conditons. It was shown in [4] that the $\theta^{\bar{a}}$-independent components of Φ transform into each other under the action of the boundary-preserving subalgebra, and can be thought of as the components of a 'reduced' superfield $\hat{\Phi}$. Boundary conditions imposed on these components will carry a representation of the reduced supersymmetry. An invariant boundary condition can be imposed by requiring that $\hat{\Phi}$ should obey *any* of a class of boundary conditions related to each other by reduced supersymmetry transformations.

We conclude by remarking that, in the case of supersymmetric quantum mechanics, there appears to be a simple short cut for finding the Nicolai map. In this case, certain combinations of the superfield components can be identified with the variables in the free bosonic theory; it has recently been observed [5] that the appropriate combinations are just the conjugate momenta of the bosonic variables in a version of the supersymmetric theory described by the modified super-Lagrangian $\mathscr{L}'$. While this observation is not yet properly understood, it is tempting to speculate that similar shortcuts might exist for more general cases.

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